Name: Feedback

SA402 · Dynamic and Stochastic Models — Fall 2023 · Uhan

Exam 1 – 9/27/2023

Instructions

- You have 50 minutes to complete this exam.
- You may use your plebe-issue TI-36X Pro calculator.
- You may not use any other materials.
- **No collaboration allowed.** All work must be your own.
- **Show all your work.** To receive full credit, your solutions must be completely correct, sufficiently justified, and easy to follow.
- Keep this booklet intact.
- **Do not discuss the contents of this exam with any midshipmen until it is returned to you.**

Problem	Weight	Score
1a	1	
1b	1	
1c	1	
2	1	
3a	1	
3b	1	
3c	1	
3d	1	
4	2	
Total		/ 100

Problem 0. Copy and sign the honor statement below. This exam will not be graded without a signed honor statement.

> The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

Signature:

Problem 1. Movr is a ride sharing service that operates in Simplexville. They have hired you to study the movement of its cars between three regions in Simplexville: Uptown, Midtown, and Downtown. The company's data analytics team has modeled the movement of a car as a Markov chain with 3 states. The states $1, 2, 3$ correspond to Uptown, Midtown, and Downtown, respectively, and each time step corresponds to one car trip. When a car reaches its destination, it stays in the destination region until it is used again. The one-step transition probability matrix for a car is:

$$\mathbf{P} = \begin{bmatrix} 0.35 & 0.47 & 0.18 \\ 0.18 & 0.63 & 0.19 \\ 0.19 & 0.52 & 0.29 \end{bmatrix}$$

Suppose at the beginning of the day, 30% of the cars are in the Uptown region, 45% in the Midtown region, and 25% in the Downtown region.

a. What is the probability that a car randomly chosen at the beginning of the day will be in the Midtown region after 6 trips? Provide your answer to 3 decimal places.

For similar examples, see Example 3 in Lesson 5, Problems 2c and 3d in the Lesson 5 Exercises, and Problem 2a in the Review Problems for Exam 1.

b. What is the probability that a car starting in the Downtown region will be in the Uptown region after 2 trips? Provide your answer to 3 decimal places.

For similar examples, see Example 2 in Lesson 5, and Problems 2b and 3c in the Lesson 5 Exercises.

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Here is the transition probability matrix for Problem 1 again:

$$\mathbf{P} = \begin{bmatrix} 0.35 & 0.47 & 0.18 \\ 0.18 & 0.63 & 0.19 \\ 0.19 & 0.52 & 0.29 \end{bmatrix}$$

c. What is the probability that a car starts in the Uptown region, stays in either the Uptown or Downtown regions for 3 trips, and then goes to the Midtown region in the 4th trip? Provide your answer to 3 decimal places.

For similar problems, see Example 4 in Lesson 5, and Problems 1c, 2d, and 3e in the Lesson 5 Exercises.

Problem 2. You have just been hired as an analyst in the Cauchy County Department of Health and Human Services. Your predecessor developed a model of the county population, in which each citizen can be classified as living in one of three location types: urban, rural, or suburban. In their model, the state of the system is defined as a citizen's current location type, and the time step is defined to be 1 year.

Describe what assumptions need to be made in order for the Markov property to hold. (You do not need to discuss whether these assumptions are realistic.)

For similar problems, see Example 1 in Lesson 7, Problem 1 in the Lesson 7 Exercises, and Problem 4 in the Review Problems for Exam 1.

Problem 3. An autonomous UUV has been programmed to move randomly between 6 regions according to a Markov chain, in which each region corresponds to a state, and each time step corresponds to one movement of the UUV. Looking at the documentation written by the programmer, you find the following one-step transition matrix:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0 & 0.25 & 0.65 & 0 & 0 & 0.10 \\ 0 & 0.65 & 0.20 & 0 & 0 & 0.15 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.20 & 0.10 & 0 & 0 & 0.70 \end{bmatrix}$$

There are two recurrent classes: $\mathcal{R}_1 = \{2, 3, 6\}$ and $\mathcal{R}_2 = \{5\}$.

a. Classify each of the 6 states as transient or recurrent. No explanation necessary.

Note that the problem gives you the two recurrent classes in this Markov chain. See the bottom of page 3 in Lesson 6 for guidance on how to use this information to classify the states as transient or recurrent.

b. Suppose the UUV reaches region 3. What is the long-run fraction of time it spends in region 3? Provide your answer to 3 decimal places.

For similar examples, see Example 5 in Lesson 6, Problems 4, 5, and 6c in the Lesson 6 Exercises, and Problem 2c in the Review Problems for Exam 1.

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Here is the transition probability matrix for Problem 3 again:

$$\mathbf{P} = \begin{bmatrix} 0.10 & 0.25 & 0.15 & 0.10 & 0.05 & 0.35 \\ 0 & 0.25 & 0.65 & 0 & 0 & 0.10 \\ 0 & 0.65 & 0.20 & 0 & 0 & 0.15 \\ 0.20 & 0.10 & 0.40 & 0.10 & 0.15 & 0.05 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0.20 & 0.10 & 0 & 0 & 0.70 \end{bmatrix}$$

c. Suppose the UUV starts in region 4. What is the probability that the UUV eventually ends up in region 5? Provide your answer to 3 decimal places.

See Example 6 in Lesson 6, as well as Problem 3 in the Lesson 6 Exercises, for similar examples.

d. What is the expected time to absorption from state 4? Provide your answer to 3 decimal places.

See Example 7 in Lesson 6, as well as Problem 3 in the Lesson 6 Exercises, for similar examples.

Problem 4. Arrow Advertising wants to study how consumers behave in its direct marketing campaign for Primal Pralines. According to historical data, the probability that a customer will purchase a box of pralines in response to a "remarketing" phone call depends on the number of months have passed since the customer last purchased a box.

If one month has passed since the customer's last purchase, the customer will purchase another box 70% of the time. If two months have passed, the customer will purchase another box 50% of the time. If three months have passed, the customer will purchase another box 30% of the time. Finally, once four months have passed, the firm will stop all future efforts to "remarket" to the customer.

One month has passed since last purchase for 40% of the customers involved. Two months have passed for 30%, and three months have passed for 20%. Finally, four months or longer have passed for 10% of the customers involved.

A customer purchases at most one box of pralines per month.

Model this setting as a Markov chain by defining:

- the state space and the meaning of each state,
- the meaning of one time step in the setting's context,
- the one-step transition probabilities, and
- the initial state probabilities.

Specify the one-step transition probabilities as a matrix, and the initial state probabilities as a vector.

- Some of you defined the state space using tuples representing purchases over consecutive months, similar to Example 2 in Lesson 7. It turns out that this is more complicated than necessary for this problem.
- Instead, try defining the following state space: $\mathcal{M} = \{1, 2, 3, 4\}$, where

$$\text{state } i \leftrightarrow \begin{cases} i \text{ months have passed since last purchase} & \text{if } i = 1, 2, 3 \\ 4 \text{ months or longer have passed since last purchase (no longer remarketed to)} & \text{if } i = 4 \end{cases}$$